2024 CHANTILLY MATH COMPETITION ELEMENTARY SCHOOL DIVISION SOLUTIONS

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Participant Information

(a) Participant Name

(b) Participant Grade Level

(c) School Name

RULES/INFORMATION

- Participants will have 100 minutes for the exam.
- Outside resources such as calculators, mobile devices, textbooks are not allowed.
- Collaboration is not allowed.
- This exam consists of 25 free response questions.
- The problems will be in order of increasing difficulty, but you may occasionally find some later questions easier, depending on experience.
- Questions 1-8 will be worth 5 points each, 9-16 worth 6 points each, and 17-25 worth 8 points each.
- The answers to the all the problems are guaranteed to be non-negative integers (0, 1, 2, ...).
- Miscellaneous - The area of a circle is $\pi \cdot r^2$ and the circumference is $2 \cdot \pi \cdot r$, where r is the radius of the circle.

QUESTIONS 1-8

Question 1

Find the tens digit of $1234321 - 12321 - 121 - 1$

Arsenii Zharkov

$1234321 - 12321 - 121 - 1 \equiv 21 - 21 - 21 - 1 \equiv -22 \equiv 78 \pmod{100}$ so the answer is $\boxed{7}$.

Question 2

How many diagonals does an octagon have?

Aryan Raj

$$\frac{8 \cdot 5}{2} = \boxed{20}.$$

Question 3

Saul and Paul play catch every day. Today, Saul brought the ball. Saul and Paul passed the ball back and forth a lot. Then, Paul went home with the ball. If Saul made 10 passes to Paul, how many passes did Paul make to Saul?

Aryan Raj

$$10 - 1 = \boxed{9}.$$

Question 4

How many 3-digit numbers are divisible by 13?

Aryan Raj

$$\lfloor \frac{999}{13} \rfloor - \lfloor \frac{99}{13} \rfloor = 76 - 7 = \boxed{69}.$$

Question 5

How many square inches are in 3 square feet?

Aryan Raj

$$3 \text{ ft}^2(\frac{12 \text{ in}}{\text{ft}})^2 = 3 \cdot 12^2 \text{ in}^2 = \boxed{432} \text{ in}^2.$$

Question 6

A man with height 6 feet stands in front of a streetlight with height 10 feet and casts a shadow of length 12 feet. In feet, how far away is the man from the streetlight?

Aryan Raj

Let the man be x feet away from the streetlight. Similar triangles gives

$$\frac{6}{12} = \frac{10}{x+12}$$

so $x + 12 = 20$ so $x = \boxed{8}$.

Question 7

How many positive integers are less than 4 times the sum of their digits?

Aryan Raj

We can rewrite this problem as the number of solutions to

$$a + 10b + 100c + 1000d + \cdots < 4a + 4b + 4c + 4d + \ldots$$

where $0 \leq a, b, c, d, \cdots \leq 9$.
This simplifies to $6b + 96c + 996d + \cdots < 3a$, so we must have $c, d, \cdots = 0$. Then, we just want the number of solutions to $2b < a$ where $0 \leq a, b \leq 9$. By casework on b, this is $9 + 7 + 5 + 3 + 1 = \boxed{25}$.

Question 8

Find the smallest prime number that cannot be written as the sum of two squares.

Aryan Raj

$2 = 1^2 + 1^2$. However, since 2 and 3 are both not perfect squares, the answer is $\boxed{3}$.

QUESTIONS 9-16

Question 9

Arsenii, Aryan, and Madhavan are writing problems for the Chantilly Math Competition. Aryan takes 5 minutes to write a math problem. Arsenii takes 12 minutes to write a math problem. Madhavan takes 15 minutes to write a math problem. How many minutes would it take for the group to write a total of 42 math problems?

Aryan Raj

$$\frac{42}{\frac{1}{5}+\frac{1}{12}+\frac{1}{15}} = \boxed{120}.$$

Question 10

Shubham is baking cookies. He wants all his cookies to be circles with radius 2 inches. Shubham has a circular dough with radius 6 inches. After cutting out the maximum number of 2 inch cookies from this circle, he continues repeating this process until he can't cut out any more cookies. How many cookies did he cut out?

Aryan Raj

$$(\frac{6}{2})^2 = \boxed{9}.$$

Question 11

Arsenii has 2024 pencils and 20 boxes. If Arsenii puts each of the 2024 pencils into one of the 20 boxes, what is the maximum number of pencils that can be in the box with the least number of pencils?

Aryan Raj

$$\lfloor \frac{2024}{20} \rfloor = \boxed{101}.$$

Question 12

Call a positive integer "sigma" if it has at least 2 digits and is divisible by the sum of its digits. What is the greatest common divisor of all "sigma" numbers?

Aryan Raj

$$\gcd(10, 21) = \boxed{1}.$$

Question 13

Madhavan is a baller. In basketball, every shot made is either 2 points or 3 points. If Madhavan made 10 shots and scored 24 points, how many 3-pointers did he make?

Aryan Raj

If Madhavan made all 2 pointers he would've gotten 20 points. Every 2-pointer that gets switched to a 3-pointer adds 1 point to the total. Therefore, the answer is $24 - 20 = \boxed{4}$.

Question 14

A rectangular garden has a length that is 4 times its width. If the perimeter of the garden is 90 meters, what is the area of the garden in square meters?

Aryan Raj

Let w be the width. Then, $2(w + 4w) = 90$ so $w = 9$ so the area is

$$9 \cdot (4 \cdot 9) = \boxed{324}.$$

Question 15

Peter and George are sharing two bags of chips. They take turns both eating some positive number chips from one of the bags. At the start, one of the bags has 20 chips and the other has 24 chips. If Peter is the first one to eat chips, how many chips should Peter eat at first to guarantee that he eats the last chip? (You may first want to determine a strategy to do this, as they can both eat as many as they want on each turn)

Aryan Raj

If Peter eats 4 chips from the bag with 24 chips, then the bags both have 20 chips. Then, Peter can just copy George to eat the last chip. Therefore, the answer is $\boxed{4}$.

Question 16

John is running a lemonade stand. For each sale, he earns $\$3$. However, whenever he has enough money, he spends $\$7$ on his favorite candy. What's the minimum combined number of sales and candy bought if he has $\$1$ remaining after buying the candy?

Aryan Raj

We want to minimize $x+y$ where $x, y \geq 0$ are positive integers such that $3x-7y=1$. This occurs at $(x,y)=(5,2)$ so the answer is $5+2=\boxed{7}$.

QUESTIONS 17-20

Question 17

George is a farmer that wants to construct an enclosure for his sheep. He makes a rectangular fence with a total perimeter of 200 ft. What's the maximum area his fence can enclose with 200 ft. of fence?

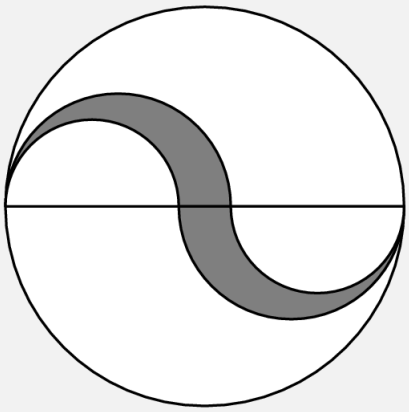
Arsenii Zharkov

$$(\frac{200}{4})^2=\boxed{2500}.$$

Question 18

If the area of the shaded region is $k \cdot \pi$ (some multiple of π), find k

The outer circle has radius 23. The shaded region is outlined by half circles whose radii are 10 and 13 and whose centers lie on the diameter of the big circle. Find the area of the shaded region.



Aryan Raj

$$13^2 - 10^2 = \boxed{69}.$$

Question 19

2024 people, numbered $1 - 2024$, sit in a circle, and there is a spoon between every pair of two adjacent people. In increasing numerical order, each person reaches for either their left or their right, each with a probability of $1/2$. Then, if there is a spoon on the side that they reached for, they take it, otherwise they do nothing. On average, how many people will get to pick up a spoon?

Shubham Patel

By linearity of expectation, the answer is $\frac{2024}{2} = \boxed{1012}$.

Question 20

Tyler wrote the numbers from 1 to 20 on a board but realized he missed one. The sum of all the remaining numbers on the board turned out to be a prime number. What is the largest number he could have missed?

Aryan Raj

The sum is $\frac{20 \cdot 21}{2} = 210$. The smallest prime above 190 is 191 so the answer is $210 - 191 = \boxed{19}$.

ADVANCED PROBLEMS

Question 21

Rishi bought some fancy chalk and wants to test it on his blackboard by playing a game. He starts by writing the integers from 0 to 5 inclusive on his blackboard. Every minute, he'll pick two of the numbers, and replace them with the sum of their sum and their product. What is the minimum possible value of the sum of the numbers on his blackboard after 5 minutes?

Aryan Raj

We can replace a and b with $ab+a+b$ and $(a+1)(b+1) = ab+a+b+1$. The product of 1 more than the numbers is constant. Therefore, our answer is $1\times2\times3\times4\times5\times6-1 = \boxed{719}$.

Question 22

how many perfect squares are in the infinite arithmetic series $3, 7, 11, 15, \ldots$?

Aryan Raj

$x^2 \equiv 3 \pmod 4$ has $\boxed{0}$ solutions.

Question 23

let $ABCD$ be a quadrilateral such that $AB = 15, BC = 20, CD = 24, DA = 7$, what is $AC + BD$?

Aryan Raj

By Pythagorean Theorem, $AC = 25$. By Ptolemy's Theorem, $AC\cdot BD = AB\cdot CD+BC\cdot DA$ so $25BD = 360 + 140 = 500$ so $BD = 20$ so the answer is $20 + 25 = \boxed{45}$.

Question 24

Find the sum of all 3-digit palindromes (the same written forwards and backwards) that are divisible by 5.

Aryan Raj

Clearly they are 505, 515, ..., 585, 595 and their sum is $550 \times 10 = \boxed{5500}$.

Question 25 continued on next page...

Question 25

We call a positive integer "skibidi" if the product of its digits is prime. How many positive integers less than 123456789 are "skibidi"?

Aryan Raj

Clearly every "skibidi" number has all digits 1 except for 1 prime digit. Since the prime digits are 2, 3, 5, and 7, there are $4n$ n-digit numbers that are "skibidi". This means there are

$$4(1+2+...+8+9) = 4\sum_{n=1}^{9} n = 4\frac{9(10)}{2} = 180$$

"skibidi" numbers with at most 9 digits. However, we must exclude the "skibidi" numbers that are at least 123456789, and there are $4+3=7$. Therefore, the answer is $180-7=\boxed{173}$.