2024 CHANTILLY MATH COMPETITION
MIDDLE SCHOOL DIVISION SOLUTIONS

DO NOT OPEN THIS PACKET UNTIL YOU ARE INSTRUCTED TO DO SO

Participant Information

(a) Participant Name

(b) Participant Grade Level

(c) School Name

RULES/INFORMATION

- Participants will have 100 minutes for the exam.
- Outside resources such as calculators, mobile devices, textbooks are not allowed.
- Collaboration is not allowed.
- This exam consists of 25 free response questions.
- The problems will be in order of increasing difficulty, but you may occasionally find some later questions easier, depending on experience.
- Questions 1-8 will be worth 5 points each, 9-16 worth 6 points each, and 17-25 worth 8 points each.
- The answers to the all the problems are guaranteed to be non-negative integers (0, 1, 2, ...).
- Miscellaneous - The volume of a cone is $\pi r^3 h/3$, where r is the radius of the cone's base and h is the height.

QUESTIONS 1-8

Question 1

Arsenii, Aryan, and Madhavan are writing problems for the Chantilly Math Competition. Arsenii takes 5 minutes to write a math problem. Aryan takes 12 minutes to write a math problem. Madhavan takes 15 minutes to write a math problem. How many minutes would it take for them to write a total of 42 math problems?

Aryan Raj

$$\frac{42}{\frac{1}{5}+\frac{1}{12}+\frac{1}{15}} = \boxed{120}.$$

Question 2

George is a farmer that wants to construct an enclosure for his sheep. He makes a rectangular fence with a total perimeter of 200 ft. What's the maximum area his fence can enclose with 200 ft. of fence?

Arsenii Zharkov

$$(\frac{200}{4})^2 = \boxed{2500}.$$

Question 3

How many positive integers are less than 4 times the sum of their digits?

Aryan Raj

We can rewrite this problem as the number of solutions to

$$a + 10b + 100c + 1000d + \cdots < 4a + 4b + 4c + 4d + \ldots$$

where $0 \leq a, b, c, d, \cdots \leq 9$.
This simplifies to $6b + 96c + 996d + \cdots < 3a$, so we must have $c, d, \cdots = 0$. Then, we just want the number of solutions to $2b < a$ where $0 \leq a, b \leq 9$. By casework on b, this is $9 + 7 + 5 + 3 + 1 = \boxed{25}$.

Question 4

Call a positive integer "sigma" if it has at least 2 digits and is divisible by the sum of its digits. What is the greatest common divisor of all "sigma" numbers?

Aryan Raj

$$\gcd(10, 21) = \boxed{1}.$$

Question 5

Anthony currently has the number 2024 on his calculator. Every time Anthony presses the button with the square root symbol on it, the number of the calculator gets replaced with its square root. How many times does Anthony have to press the button before the number on the calculator becomes less than 2?

Aryan Raj

$[1, 2)$ takes 0 presses
$[2, 2^2)$ takes 1 press
$[2^2, 2^4)$ takes 2 presses
$[2^4, 2^8)$ takes 3 presses
$[2^8, 2^{16})$ takes 4 presses
Since $2024 \in [2^8, 2^{16})$, the answer is $\boxed{4}$.

Question 6

A rectangular garden has a length that is 4 times its width. If the perimeter of the garden is 90 meters, what is the area of the garden in square meters?

Aryan Raj

Let w be the width. Then, $2(w + 4w) = 90$ so $w = 9$ so the area is

$$9 \cdot (4 \cdot 9) = \boxed{324}.$$

Question 7

Arsenii has 2024 pencils and 20 boxes. If Arsenii puts each of the 2024 pencils into one of the 20 boxes, what is the maximum number of pencils that can be in the box with the least number of pencils?

Aryan Raj

$$\lfloor \frac{2024}{20} \rfloor = \boxed{101}.$$

Question 8 continued on next page...

Question 8

Tyler wrote the numbers from 1 to 20 on a board but realized he missed one. The sum of all the remaining numbers on the board turned out to be a prime number. What is the largest number he could have missed?

Aryan Raj

The sum is $\frac{20 \cdot 21}{2} = 210$. The smallest prime above 190 is 191 so the answer is $210 - 191 = \boxed{19}$.

QUESTIONS 9-16

Question 9

Kylor is drinking lemonade from a cup. Currently, the lemonade is 40% lemon juice and 60% water. Kylor drinks 60% of the lemonade but then decides it is too sour, so he fills the rest of the cup with water and stirs it. Then, Kylor drinks 25% of the lemonade before deciding it is too bland, so he fills the rest of the cup with lemon juice and stirs it. Kylor takes a sip of the lemonade and decides it is just right. What is the percentage of the lemonade is lemon juice?

Aryan Raj

WLOG, start with 40 lemon juices and 60 waters. After drinking 60%, there are 16 lemon juices and 24 waters. After filling rest with water, there are 16 lemon juices and 84 waters. After drinking 25%, there are 12 lemon juices and 63 waters. After filling rest with lemon juice, there are $\boxed{37}$ lemon juices and 63 waters.

Question 10

Let $ABCD$ be a quadrilateral such that $AB = 15, BC = 20, CD = 24, DA = 7$, what is $AC + BD$?

Aryan Raj

By Pythagorean Theorem, $AC = 25$. By Ptolemy's Theorem, $AC \cdot BD = AB \cdot CD + BC \cdot DA$ so $25BD = 360 + 140 = 500$ so $BD = 20$ so the answer is $20 + 25 = \boxed{45}$.

Question 11

2024 people, numbered $1-2024$, sit in a circle, and there is a spoon between every pair of two adjacent people. In increasing numerical order, each person reaches for either their left or their right, each with a probability of $1/2$. Then, if there is a spoon on the side that they reached for, they take it, otherwise they do nothing. On average, how many people will get to pick up a spoon?

Shubham Patel

By linearity of expectation, the answer is $\frac{2024}{2} = \boxed{1012}$.

Question 12

how many perfect squares are in the infinite arithmetic series $3, 7, 11, 15, \ldots$?

Aryan Raj

$x^2 \equiv 3 \pmod 4$ has $\boxed{0}$ solutions.

Question 13

If $x + 1/x = 3$, what is $x^3 + 1/x^3$?

Aryan Raj

$$x^3 + \frac{1}{x^3} = (x + \frac{1}{x})^3 - 3(x + \frac{1}{x}) = 3^3 - 3 \cdot 3 = \boxed{18}.$$

Question 14

If the smallest number of minutes after midnight where the hands make a 60° angle is m/n for relatively prime m and n (m/n in most simplified form), find $m + n$.

Aryan Raj

Every minute, the minute hand turns 6° and the hour hand turns $.5^\circ$ so we get $\frac{60}{6-.5} = \frac{120}{11}$ for an answer of $120 + 11 = \boxed{131}$.

Question 15

Bob rolls two fair many-sided dice, the first labeled with the numbers $1, 2, \ldots, 49$ and the second labeled with the numbers $1, 2, \ldots, 169$. Find the average value of the sum of the two die faces that land on top.

Shubham Patel

$$\mathbb{E}[a+b] = \mathbb{E}[a] + \mathbb{E}[b] = \frac{1+49}{2} + \frac{1+169}{2} = 25 + 85 = \boxed{110}.$$

Question 16

What is the remainder when 2^{2024} is divided by 13?

Aryan Raj

By Fermat's Little Theorem,

$$2^{2024} \equiv 2^8 \equiv 256 \equiv \boxed{9} \pmod{13}.$$

QUESTIONS 17-20

Question 17

How many positive integers are less than 4 times the sum of their digits?

Aryan Raj

We can rewrite this problem as the number of solutions to

$$a + 10b + 100c + 1000d + \cdots < 4a + 4b + 4c + 4d + \ldots$$

where $0 \leq a, b, c, d, \cdots \leq 9$.
This simplifies to $6b + 96c + 996d + \cdots < 3a$, so we must have $c, d, \cdots = 0$. Then, we just want the number of solutions to $2b < a$ where $0 \leq a, b \leq 9$. By casework on b, this is $9 + 7 + 5 + 3 + 1 = \boxed{25}$.

Question 18

John is planning a circular flower garden with a concentric walkway around it. The inner circular garden has a radius of 4 feet. The walkway around has a uniform width of 3 feet. If bags of concrete cover 4π square feet each, how many total bags should John buy to completely fill in the garden walkway area?

Anirudh Gudishetty

Step 1: Find the Inner Garden Circle Area

$$\begin{aligned}\text{Radius} &= 4 \text{ feet} \\ \text{Area} &= \pi r^2 = \pi(4^2) = 16\pi \text{ ft}^2\end{aligned}$$

Step 2: Find the Outer Circle Area (Total Area)

$$\begin{aligned}\text{Outer Radius} &= \text{Inner Radius} + \text{Walkway Width} = 4 + 3 = 7 \text{ feet} \\ \text{Outer Area} &= \pi r^2 = \pi(7^2) = 49\pi \text{ ft}^2\end{aligned}$$

Step 3: Find the Walkway Area

$$\begin{aligned}\text{Walkway Area} &= \text{Total Outer Area} - \text{Inner Area} \\ &= 49\pi - 16\pi \\ &= 33\pi \text{ ft}^2\end{aligned}$$

Step 4: Find the Total Number of Bags
Each bag of concrete covers 4π square feet. The number of bags needed is:

$$\begin{aligned}\text{Bags Needed} &= \frac{\text{Walkway Area}}{\text{Bag Coverage}} \\ &= \frac{33\pi}{4\pi} \\ &= \frac{33}{4} \\ &\approx 8.25\end{aligned}$$

Since John cannot purchase a fraction of a bag, he needs to round up to $\boxed{9}$ bags to fully cover the walkway.

Question 19

Jeffrey is in prison so he is only allowed to go outside on Sundays. On Friday, Jeffrey heard that there was a 70% chance that it would rain on Saturday. He also heard that if it rained on Saturday, there was a 25% chance that it would rain on Sunday. However, if it didn't rain on Saturday, there was a 40% chance that it would rain on Sunday. Jeffrey went outside on Sunday and was very disappointed because it was raining. If the probability it was raining on Saturday can be expressed as m/n for relatively prime positive integers m and n, compute $m+n$.

Aryan Raj

$$\frac{.7 \cdot .25}{.7 \cdot .25 + .3 \cdot .4} = \frac{35}{59}$$

so the answer is $35 + 59 = \boxed{94}$.

Question 20

The factorial number n is the product $1 \cdot 2 \cdots (n-1) \cdot n$. What is the smallest positive integer n such that $10! \cdot n$ is a perfect cube?

Aryan Raj

$10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, so the answer is $n = 2 \cdot 3^2 \cdot 5 \cdot 7^2 = \boxed{4410}$.

ADVANCED PROBLEMS

Question 21

Shawn is bored so he flips a coin over and over. If the probability that he flips 3 heads in a row before he flips 2 tails in a row is m/n for relatively prime positive integers m and n, find $m+n$.

Aryan Raj

Let $P(x)$ be the probability that Shawn flips 3 heads in a row before he flips 2 tails in a row if he just flipped x heads in a row. $P(0)$ means Shawn just flipped a tail. $P(-1)$ is the initial condition where Shawn hasn't flipped anything yet, AKA our answer.

$$\begin{aligned} P(3) &= 1 \\ P(2) &= \frac{P(3)}{2} + \frac{P(0)}{2} = \frac{1}{2} + \frac{P(0)}{2} \\ P(1) &= \frac{P(2)}{2} + \frac{P(0)}{2} = \frac{1}{4} + \frac{3P(0)}{4} \\ P(0) &= \frac{P(1)}{2} = \frac{1}{8} + \frac{3P(0)}{8} \end{aligned}$$

so $P(0) = \frac{1}{5}$ and $P(1) = \frac{2}{5}$. $P(-1) = \frac{P(0)}{2} + \frac{P(1)}{2} = \frac{3}{10}$ so our answer is $\boxed{13}$.

Question 22

Let there be a triangle bounded by the lines $y = 4-2x, y = x-2$, and $y = 4-x$. A solid is then formed by revolving this triangle around the y-axis. If the volume of this solid can be expressed as $k \cdot \pi$ for some positive integer k, find k.

Aryan Raj

Add and subtract various cones to get $k = \boxed{18}$.

Question 23

Suppose $6^x = 11$ and $11^y = 216$. What is xy?

Almas Zhorayev

$6^{xy} = (6^x)^y = 11^y = 216$ so $xy = \log_6 216 = \boxed{3}$.

Question 24 continued on next page...

Question 24

Find the sum of all integers x such that

$$\frac{2^{x-14}}{x-13}$$

is prime.

Shubham Patel

The only value of x that satisfies the given condition is $x = 17$, hence, the answer is $\boxed{17}$.

Question 25

We call a positive integer "skibidi" if the product of its digits is prime. How many positive integers less than 123456789 are "skibidi"?

Aryan Raj

Clearly every "skibidi" number has all digits 1 except for 1 prime digit. Since the prime digits are 2, 3, 5, and 7, there are $4n$ n-digit numbers that are "skibidi". This means there are

$$4(1+2+...+8+9) = 4\sum_{n=1}^{9} n = 4\frac{9(10)}{2} = 180$$

"skibidi" numbers with at most 9 digits. However, we must exclude the "skibidi" numbers that are at least 123456789, and there are $4+3=7$. Therefore, the answer is $180-7=\boxed{173}$.