2025 Chantilly Math Competition
Elementary School Division Solutions

1. (5 pts) Tyler has 23 candies on Halloween. Francis has 37 candies on Halloween. Tyler is sad because he has less candies. How many candies should Francis give to Tyler so that they both have the same number of candies?

   Aryan Raj

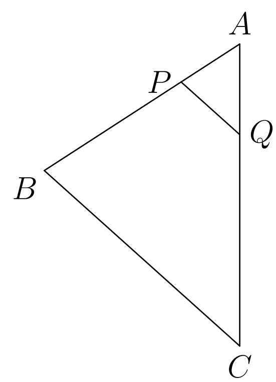
   Solution
   $$\frac{37-23}{2}=\boxed{7}$$

2. (5 pts) Neeraj has a pair of red socks, a pair of green socks, and a pair of blue socks, for a total of 6 socks. If Neeraj randomly picks 2 socks out of the 6 to wear, the probability that they are the same color can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $m+n$.

   Aryan Raj

   Solution No matter what the first sock is, we will have 5 socks left, and 1 of them will match the first sock, so our desired probability is $\frac{1}{5}$, for an answer of $1+5=\boxed{6}$.

3. (5 pts) In the below diagram, $\angle BAC = 57°$, $\angle PQA = 48°$, and PQ is parallel to BC. Compute the degree measure of $\angle ABC$.



   Victor Moldoveanu

   Solution
   $$\angle ABC = 180° - \angle BAC - \angle BCA = 180° - 57° - \angle PQA = 180° - 57° - 48° = \boxed{75}°.$$

4. (5 pts) How many distinct prime divisors does 360 have?

   Victor Moldoveanu

   Solution $360 = 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 5$ so the answer is $|\{2,3,5\}| = \boxed{3}$.

5. (6 pts) Compute 28% of 25.

   Victor Moldoveanu

   Solution
   $$28\% \text{ of } 25 = \frac{28}{100}\cdot 25 = \frac{28}{4} = \boxed{7}.$$

6. (6 pts) In how many ways can the letters in `CHANTILLY` be arranged such that there are two letters between `C` and `T`?

   Shubham Patel

   Solution

$$(9-3)\cdot 2\cdot \frac{7!}{2} = 6\cdot 7! = 6\cdot 5040 = \boxed{30240}.$$

7. (6 pts) A circle has a radius of 5. A chord of the circle is 8 units long. How far is the chord from the center of the circle?

   Aryan Raj

   Solution Drop an altitude from the circle's center to the chord. Half of the chord has length 4, so by Pythagorean Theorem, the length of the altitude is

$$\sqrt{5^2-4^2} = \boxed{3}.$$

8. (6 pts) Which digit X makes $\overline{12X4567}$ divisible by 9?

   Note that $\overline{d_1d_2\dots d_n}$ represents the concatenation of digits $d_1, d_2, \dots, d_n$.

   Vikhyaath Ale

   Solution Since $10^k \equiv 1^k \equiv 1 \pmod 9$,

$$\overline{12X4567} \equiv 1+2+X+4+5+6+7 \equiv X-2 \pmod 9,$$

   so $X = \boxed{2}$.

9. (7 pts) Aarush is counting backwards by 8. His first three numbers are $1000, 992, 984$. What is his 25th number?

   Aarush Nutalapati

   Solution Aarush's counting sequence is arithmetic with first term $a_1 = 1000$ and common difference $d = -8$. The n-th term is

$$a_n = a_1 + (n-1)d.$$

   For the 25th term ($n = 25$):

$$a_{25} = 1000 + (25-1)(-8) = 1000 - 192 = \boxed{808}.$$

10. (7 pts) Consider a 3×5 grid (15 unit squares) and seven 1×2 dominoes, each fully covering exactly two adjacent unit squares. We wish to tile the grid with these dominoes so that the dominoes do not overlap and the only uncovered unit square is a corner of the grid.

    Determine the number of distinct tilings that satisfy these conditions. Note that the order in which the dominoes are placed does not matter.

    Aryan Raj

    Solution WLOG let the untiled corner be in the top left. Let $f(n)$ be the answer for a $3 \times n$ rectangle with its top left corner gone. Let $g(n)$ be the answer for a $3 \times n$ rectangle where n is even.

    $f(5) = f(3) + g(4)$

    $f(3) = 1 + g(2)$

    $g(2) = 3$ by inspection

    $f(3) = 4$

    $g(4) = 2f(3) + g(2) = 8 + 3 = 11$

    $f(5) = 4 + 11 = 15$

    Since there are 4 corners, the answer is $4f(5) = \boxed{60}$.

11. (7 pts) A square and a circle have the same perimeter. If the side of the square is 4 units, what is the area of the circle to the nearest integer?

    Aryan Raj

    Solution The circle has circumference $4 \cdot 4 = 16$ so its radius is $\frac{16}{2\pi} = \frac{8}{\pi}$ so its area is

    $$\pi(\frac{8}{\pi})^2 = \frac{64}{\pi} \approx \boxed{20}.$$

12. (7 pts) Nischal thinks of a 3-digit positive integer N satisfying the following 3 properties:

    i. It is a palindrome (reads the same forwards and backwards).
    ii. The sum of its digits is 10.
    iii. It is divisible by 5.

    Find N.

    Shubham Patel

    Solution Let the N be represented as $\overline{aba}$, where a and b are digits. Since N is divisible by 5, $a = 5$ ($a \neq 0$ because N is a three-digit number). Now, because the sum of the digits of N is 10, we have $2a + b = 10 \implies b = 0$. Thus, $N = \boxed{505}$.

13. (8 pts) What is the largest possible product of positive integers which sum to 16?

    Aryan Raj

    Solution We see that $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$ and $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil \geq n$ for $n \geq 4$, so having any positive integers at least 4 is suboptimal.

    We also see that $1 \cdot a < 1 + a$ for all a, so we should not have 1, and only have 2 and 3.

    Therefore, the answer is $\max(2^8, 2^2 \cdot 3^4) = \boxed{324}$.

14. (8 pts) Aiden has 3 dice. One die has 4 sides numbered $1, 2, 3, 4$. One die has 6 sides numbered $1, 2, 3, 4, 5, 6$. One die has 8 sides numbered $1, 2, 3, 4, 5, 6, 7, 8$. Aiden picked one die out of the three uniformly at random and rolled it. The bottom face of the rolled die had the number 1 on it. If the probability that he rolled the 8-sided die can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, compute $a + 10b$.

    Aryan Raj

    Solution

$$\frac{\frac{1}{3} \cdot \frac{1}{8}}{\frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{8}} = \frac{3}{13}$$

    so the answer is $3 + 10(13) = \boxed{133}$.

15. (8 pts) What is the maximal number of spheres of radius 1 that can fit into a cube with volume 64?

    Riyan Nayak

    Solution The cube has side length $\sqrt[3]{64} = 4$, and the diameter of each sphere is $2(1) = 2$, so the answer is $(\frac{4}{2})^3 = \boxed{8}$.

16. (8 pts) A group of 4 positive integers has an equal mean and range. What is the smallest possible sum of the squares of the 4 positive integers?

    Aryan Raj

    Solution Let the 4 positive integers be $a \le b \le c \le d$. Then, $a + b + c + d = 4(d - a)$ so $5a + b + c = 3d$. Since a, b, and c are positive integers, $3d \ge 7$ so $d \ge 3$. When $d = 3$, $a^2 + b^2 + c^2 + d^2$ is clearly minimized at $1^2 + 2^2 + 2^2 + 3^2 = 18$. When $d > 3$, $d^2 \ge 16$, and $a^2 + b^2 + c^2 \ge 1 + 1 + 1 = 3$, and $16 + 3 > 18$, so the answer is $\boxed{18}$.

17. (9 pts) Sally's calculator shows the number 2. She repeatedly presses the square button, which multiplies the number on the calculator by itself. How many times must she press the square button until the number shown on the calculator has at least 2025 digits?

    Aryan Raj

    Solution Pressing the square button k times will display 2^{2^k} on the calculator. We notice that $2^{10} = 1024$, which is approximately 10^3. Therefore, 2^{6750} is slightly larger than 10^{2025}. Pressing the square button 12 times would give 2^{4096}, which is clearly too small to have at least 2025 digits. However, pressing the square button 13 times would give $2^{8192} > 2^{6750}$. Therefore, the answer is $\boxed{13}$.

18. (9 pts) Daniel holds 12 real nuggets in his hand. 3 fanum-taxers, one after another, each randomly eat 1 of the 12 nuggets in Daniel's hand and put a fake nugget in Daniel's hand. The probability that Daniel ends up with exactly 10 real nuggets is $\frac{m}{n}$ for relative prime positive integers m and n. Compute $m + n$.

    Olivia Wu

    Solution If Daniel comes back to find just 2 fake nuggets, then exactly 1 of the fanum-taxers must have randomly chosen to replace a fake nugget. If this fanum-taxer was the second fanum-taxer, then the probability of having 2 fake nuggets at the end is $(1)(\frac{1}{12})(\frac{11}{12}) = \frac{11}{144}$. If this fanum-taxer was the third fanum-taxer, then the probability is $(1)(\frac{11}{12})(\frac{2}{12}) = \frac{22}{144}$. Adding this up, we get $\frac{33}{144}$, so $m + n = 33 + 144 = \boxed{177}$.

19. (9 pts) What is the maximum number of acute angles that can be in a non-self-intersecting polygon with 2025 sides?

Aryan Raj

Solution The sum of the interior angles of the polygon is $180(2025-2) = 180 \cdot 2023$.

Let there be a acute angles in the polygon. Then,

$$90a + 360(2025-a) > 180 \cdot 2023$$

so the answer is

$$\lceil \frac{360 \cdot 2025 - 180 \cdot 2023}{270} - 1 \rceil = \boxed{1351}.$$

20. (9 pts) Let $\tau(x)$ denote the number of positive integer divisors of x. Find the sum of all positive integers n such that $n + \tau(n) = 38$.

Sujith Miriyala

Solution Since $\tau(n) \le 2\lfloor \sqrt{n} \rfloor$, we must have $n \ge 38 - 2(5) = 28$.

$\tau(n)$ is only odd when n is a perfect square, but 36 is even.

Therefore, $\tau(n)$ and n must both be even.

$28 + \tau(28) = 28 + 6 = 34$

$30 + \tau(30) = 30 + 8 = 38$

$32 + \tau(32) = 32 + 6 = 38$

$34 + \tau(34) = 34 + 4 = 38$

Therefore, the answer is $30 + 32 + 34 = \boxed{96}$.

21. (12 pts) William writes the integers from 1 to 2025 inclusive in a row on a whiteboard. Sally sums all the digits written on the whiteboard. What sum does Sally get?

Aryan Raj

Solution We will compute sum of digits for 000 to 999 inclusive. We can include leading zeros because they don't affect the sum. EV of sum of 3 digits is $3 \cdot \frac{9}{2} = \frac{27}{2}$ by Linearity of Expectation. There are 1000 of the numbers so we get 13500.

Now for 1000 to 1999 inclusive we get the sum of $13500 + 1000 = 14500$, so total sum so far is $13500 + 14500 = 28000$.

For 2000 to 2025, we get $45 \cdot 2 + 15 = 105$ from ones place. We get $10 + 2 \cdot 6 = 22$ from tens place. We get $2 \cdot 26 = 52$ from thousands place.

Answer is $28000 + 105 + 22 + 52 = \boxed{28179}$.

22. (12 pts) Let $\tau(x)$ denote the number of positive integer divisors of x. Compute the number of positive integers n such that $n \leq 100$ and $\tau(\tau(n)) = \tau(n) - 1$.

Aryan Raj

Solution $\tau(x) = x - 1$ only has solutions $x = 3$ and $x = 4$.

This is because for $x \geq 6$, $\tau(x) < 2\sqrt{x} < x - 1$.

The set of primes at most 50 is $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$.

There are $14 + 9 + 5 + 2 = 30$ semiprimes.

There are 4 squares and 2 cubes.

Therefore, the answer is $30 + 4 + 2 = \boxed{36}$.

23. (12 pts) Note that $x^5 = x \cdot x \cdot x \cdot x \cdot x$. Compute the sum of all possible integer values of

$$\frac{x^5 - 5}{x - 1},$$

where $x \neq 1$ is an integer.

Shubham Patel

Solution We see that

$$\frac{x^5 - 5}{x - 1} = x^4 + x^3 + x^2 + x + 1 - \frac{4}{x - 1},$$

so $(x - 1)|4$.

Therefore, $x \in \{-3, -1, 0, 2, 3, 5\}$, so the answer is

$$\sum_{a \in \{-3,-1,0,2,3,5\}} \frac{a^5 - 5}{a - 1} = 62 + 3 + 5 + 27 + 119 + 780 = \boxed{996}.$$

24. (12 pts) In a cafeteria, there are 19 chairs arranged in a circle. Seven children each sit on a chair such that no two children are seated adjacent to each other. Out of these children, three are wearing blue shirts, three are wearing green shirts, and one is wearing a white shirt. Assume that children wearing the same shirt are indistinguishable and that any two seating arrangements that can be rotated to form the other are considered the same. How many possible seating arrangements are there?

Sujith Miriyala

Solution To solve this, we can think about the situation in a different way. Suppose the seven children are seated on a circular arrangement of only seven chairs. The given problem is then equivalent to finding the number of ways to insert empty chairs between the children such that no two children are adjacent to each other. First, we find the amount of ways to seat the children on the seven chairs. To account for rotational symmetry, we fix the child with the white shirt at a specific chair, and count the number of orderings of the rest of the children, yielding $\binom{6}{3} = 20$. We can consider the seven "gaps" between the seven children as bins and the twelve remaining empty chairs as balls. Since no two children should be adjacent to each other, we place one empty chair in each gap, giving 5 leftover empty chairs that we can fully distribute, which yields $\binom{5+7-1}{7-1} = \binom{11}{6} = 462$. Thus, the total number of ways is $462 \cdot 20 = \boxed{9240}$.

25. (12 pts) All the words that can be formed by using all the letters in `CHANTILLY` are listed in alphabetical order. How many words are before the word `CHANTILLY`?

    Aryan Raj

    Solution `CHANTILLY` written in alphabetical order is `ACHILLNTY`.

    There are $\frac{8!}{2} = 20160$ words before `CAHILLNTY`.

    There are $\frac{7!}{2} = 2520$ more words before `CHAILLNTY`.

    There are $\frac{5!}{2} = 60$ more words before `CHALILNTY`.

    There are $5! = 120$ more words before `CHANILLTY`.

    There are $\frac{4!}{2} = 12$ more words before `CHANLILTY`.

    There are $4! = 24$ more words before `CHANTILLY`.

    The answer is $20160 + 2520 + 60 + 120 + 12 + 24 = \boxed{22896}$.

TB. (Tiebreak) Nischal has 2025 squares in a row. He wants to color each of them red, green, or blue. Estimate the sum of the digits of the number of colorings where no 3 consecutive squares are the same color.

Shubham Patel

Solution The answer is $\boxed{4083}$. Read below to see how to make an accurate estimate.

Let $a(n)$ be the answer when there are n squares instead of 2025 squares.

If the $n-1$th and nth squares are the same color, there are $a(n-2)$ ways to color the first $n-2$ squares, and 2 ways to color the $n-1$th and nth squares.

If the $n-1$th and nth squares are different colors, there are $a(n-1)$ ways to color the first $n-1$ squares, and 2 ways to color the nth square.

Therefore, $a(n) = 2a(n-1) + 2a(n-2)$.

If

$$A(x) = \sum_{n=0}^{\infty} a(n)x^n,$$

then this means that

$$A(x) = (2x + 2x^2)A(x) + \frac{3}{2},$$

so

$$A(x) = \frac{\frac{3}{2}}{1 - 2x - 2x^2} = \frac{\frac{3+\sqrt{3}}{4}}{1-(1+\sqrt{3})x} + \frac{\frac{3-\sqrt{3}}{4}}{1-(1-\sqrt{3})x}.$$

This means that,

$$a(n) = \frac{3+\sqrt{3}}{4}(1+\sqrt{3})^n + \frac{3-\sqrt{3}}{4}(1-\sqrt{3})^n \approx \frac{3+\sqrt{3}}{4}(1+\sqrt{3})^n.$$

Estimating logarithms can provide a good estimate for the number of digits of $a(2025)$, after which multiplying by $\frac{0+9}{2} = 4.5$ provides a good estimate for the sum of the digits of $a(2025)$.

For reference, $a(2025) =$ 917808118493726568895820944520127200653634968068853703533634576084510254734192634207409183715492603429742822695400665487852488949950795539267451724739405719111610226946564532925160904365615966315335896530702159276990086952374355258305480719038858991756109515182844006836691123902875906673270110393978867250869075769998500380838861275622062344372855877152595697088991956576303870902095518445225715751387535368837952916467923789417398931266174162733182715825906334885092217842425333839697036444651722263387145550767404066385487877516807289529093818006004548446181889763000699063254723864963609849494578169138600731311982237898393495546145182550284543444119791823485080216549965388471444141299403254076436332900146777140267723287800277143899730070264464329899049461142407402176623577817858842488638991117548061476729095104894281371058779927113631966245734587491823311787935809141022392\32.