2025 Chantilly Math Competition
Middle School Division Solutions

1. (5 pts) Saul's age is currently a quarter of Paul's age. In 10 years, Saul's age will be half of Paul's age. How old is Paul?

   Aryan Raj

   Solution We let Paul be x years old. Then, since Paul is currently $\frac{x}{4}$ years old, we get that $2(\frac{x}{4}+10) = x+10$, so $\frac{x}{2}+20 = x+10$, so $10 = \frac{x}{2}$, so $x = \boxed{20}$.

2. (5 pts) Ella loves bubble tea, but there are so many toppings to choose from! One serving of pearls cost \$1, pudding costs \$2, and jelly costs \$3. If Ella decides to spend exactly \$6 on toppings, how many ways can she customize her bubble tea? She can get more than one serving of any topping.

   Olivia Wu

   Solution The order of toppings does not matter, so we just need to find the different combinations of 1, 2, and 3 that sum up to 6. They are $\{3, 3\}$, $\{3, 2, 1\}$, $\{3, 1, 1, 1\}$, $\{2, 2, 2\}$, $\{2, 2, 1, 1\}$, $\{2, 1, 1, 1, 1\}$, and $\{1, 1, 1, 1, 1, 1\}$. Thus, Ella can order $\boxed{7}$ possible drinks.

3. (5 pts) A parking lot has dimensions of $20' \times 30'$. How many cars of $4' \times 3'$ can fit inside the lot if every car's long side is parallel to the parking lot's long side?

   Rudra Khanwalkar

   Solution
   $$\lfloor \frac{30}{4} \rfloor \lfloor \frac{20}{3} \rfloor = 7 \cdot 6 = \boxed{42}.$$

4. (5 pts) Five distinct positive integers are written in a line in increasing order. If the average of any 2 adjacent numbers in the line is a prime integer, compute the minimum possible sum of the five integers in the line.

   Riyan Nayak

   Solution We will use a greedy algorithm to write the 5 integers. If our current integer is a, our next integer will be $2p - a$ for the smallest prime $p > a$. This is clearly optimal.

   When starting with 1, this greedy algorithm gives $1 + 3 + 7 + 15 + 19 = 45$.

   When starting with 2, this greedy algorithm gives $2 + 4 + 6 + 8 + 14 = 34$.

   Note that all 5 integers must have the same parity in order for each average to be an integer.

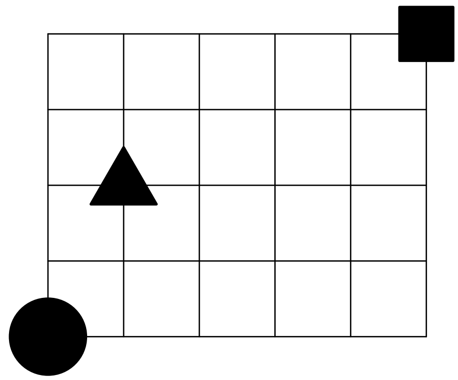
   Since $3 + 5 + 7 + 9 + 11 = 35$, this proves that the answer is $2 + 4 + 6 + 8 + 14 = \boxed{34}$.

5. (6 pts) Given that $1^x = y$, $2^y = z$, and $3^z = x$, find $x^y + y^z + z^x$.

   Krithik Manoharan

   Solution We get that $y = 1$, so $z = 2$, and therefore $x = 9$. The answer becomes
   $$9^1 + 1^2 + 2^9 = 9 + 1 + 512 = \boxed{522}.$$

6. (6 pts) The diagram below shows a 5×4 rectangular grid. Consider paths from the starting point $(0,0)$ (marked by the circle) to the destination $(5,4)$ (marked by the square). Movement along the grid lines is restricted to 1 unit rightwards or 1 unit upwards at each step. How many such paths avoid passing through the point $(1,2)$ (marked by the triangle)?



Ryan Chan

Solution

$$\binom{9}{5} - \binom{3}{1}\binom{6}{4} = \boxed{81}.$$

7. (6 pts) There exist points A, B, C, D, and E in the plane such that $AB = 75$, $BC = 72$, $CD = 16$, $DE = 12$, and $\angle ACB = \angle ACE = \angle CDE = 90°$. Find the length of segment AE.

Ryan Chan

Solution By repeated application of Pythagorean Theorem, the answer is

$$\sqrt{12^2 + 16^2 + 75^2 - 72^2} = \boxed{29}.$$

8. (6 pts) If a, b, and c are integers such that $a + b = -11$ and $bc = 20$, compute the maximum possible value of $\max(ab, bc, ca)$.

Aryan Raj

Solution ab is maximized at $(-5)(-6) = 30$ and bc is always 20. We will try to maximize ca. We see that $ca = (\frac{20}{b})(-11 - b) = -\frac{220}{b} - 20$. This is clearly maximized when $b = -1$ where we get $ca = 220 - 20 = \boxed{200}$.

9. (7 pts) Compute the sum of $|P(3)|$ over all monic cubics $P(x) = x^3 + ax^2 + bx + c$ such that

$$P(a) = P(b) = P(c) = 0.$$

Shubham Patel

Solution If the integer coefficients a, b, c of $P(x) = x^3 + ax^2 + bx + c$ are also its roots, comparing $P(x)$ with its factored form $(x - a)(x - b)(x - c)$ yields three key equations:

(a) $2a + b + c = 0$

(b) $ab + ac + bc = b$

(c) $c(ab + 1) = 0$.

From the third equation, $c(ab + 1) = 0$, either $c = 0$ or $ab = -1$.

Case 1: $c = 0$: The equations simplify to $b = -2a$ and $b(a - 1) = 0$. Substituting $b = -2a$ into the second gives $-2a(a - 1) = 0$. This implies $a = 0$ (so $b = 0$) or $a = 1$ (so $b = -2$). This yields two solutions for (a, b, c):

- $(0, 0, 0)$, for which $P_1(x) = x^3$.
- $(1, -2, 0)$, for which $P_2(x) = x^3 + x^2 - 2x$.

Case 2: $ab = -1$: Since a and b are integers, (a, b) must be $(1, -1)$ or $(-1, 1)$.

- If $a = 1, b = -1$: From $2a+b+c = 0$, we get $2(1)+(-1)+c = 0$, which means $1+c = 0$, so $c = -1$. The triplet $(1, -1, -1)$ satisfies $ab + ac + bc = b$ (as $(1)(-1) + (1)(-1) + (-1)(-1) = -1-1+1 = -1$, which is b). This gives the polynomial $P_3(x) = x^3+x^2-x-1$.
- If $a = -1, b = 1$: This leads to $c = 1$. However, the triplet $(-1, 1, 1)$ does not satisfy $ab+ac+bc = b$ (as $(-1)(1)+(-1)(1)+(1)(1) = -1-1+1 = -1 \neq 1$). This case yields no solution.

Thus, the valid (a, b, c) sets are $(0, 0, 0)$, $(1, -2, 0)$, and $(1, -1, -1)$. The three polynomials are $P_1(x) = x^3$, $P_2(x) = x^3 + x^2 - 2x$, and $P_3(x) = x^3 + x^2 - x - 1$. Evaluating them at $x = 3$:

- $P_1(3) = 3^3 = 27$.
- $P_2(3) = 3^3 + 3^2 - 2(3) = 27 + 9 - 6 = 30$.
- $P_3(3) = 3^3 + 3^2 - 3 - 1 = 27 + 9 - 3 - 1 = 32$.

The sum of the absolute values of these evaluations is $|27| + |30| + |32| = 27 + 30 + 32 = \boxed{89}$.

10. (7 pts) John writes all distinct permutations of the string `JJOOHHNN` on his whiteboard. How many permutations on his whiteboard contain the substring `JOHN`?

Riyan Nayak

Solution We must have one set of `JOHN` and scramble the rest. Call the block of `JOHN` the block `X`. We must have permutations of `J`,`O`,`H`,`N`, and `X`. This gives $5! = 120$ cases. However, we overcounted by 1 (`JOHNJOHN` is counted twice) so subtract 1 and get $\boxed{119}$.

11. (7 pts) Let $MONKEY$ be a hexagon and CAT be a triangle inside $MONKEY$ such that $MOAT$, $CANK$, and $YECT$ are all squares. If CAT is an equilateral triangle with side length 2, then the area of $MONKEY$ can be expressed as $\sqrt{a}+\sqrt{b}$ for positive integers a and b. Compute $a+b$.

Aryan Raj

Solution

$$\frac{2^2\sqrt{3}}{4}+3\cdot 2^2+\frac{(2\sqrt{3})^2\sqrt{3}}{4}=12+4\sqrt{3}=\sqrt{144}+\sqrt{48},$$

so the answer is $144+48=\boxed{192}$.

12. (7 pts) Compute the number ordered triples of positive integers (x,y,z) that satisfy the equation $x^1y^2z^3=2025$.

Shubham Patel

Solution First, we prime factorize 2025, yielding $x^1y^2z^3=3^4{\cdot}5^2$. Now, we make the following subsitutions:

$$x=3^a\cdot 5^b$$
$$y=3^c\cdot 5^d$$
$$z=3^e\cdot 5^f$$

Plugging these in, we get $(3^a\cdot 5^b)^1(3^c\cdot 5^d)^2(3^e\cdot 5^f)^3$. So, $a+2c+3e=4$ and $b+2d+3f=2$. For the second equation, we have 2 solutions: $(2,0,0)$ and $(0,1,0)$. For the first equation, we have the 4 solutions: $(0,2,0)$, $(1,0,1)$, $(4,0,0)$, and $(2,1,0)$. Hence, there are $4\cdot 2=\boxed{8}$ solutions.

13. (8 pts) Let S be the sum of the real solutions to

$$x^4+6x^3-3x^2+6x+1=0$$

If $S=a+b\sqrt{c}$ for integers a, b, and c where c is positive and square-free, find $|a|+|b|+|c|$.

Olivia Wu

Solution Observe that $x=0$ is not a solution, so we can divide the equation by x^2 to get

$$x^2+6x-3+\frac{6}{x}+\frac{1}{x^2}=0$$

Let $y=x+\frac{1}{x}$. Note that x is real when $y^2\geq 4$.
Then $x^2+\frac{1}{x^2}=y^2-2$. Thus, we have

$$y^2-2+6y-3=0$$

Completing the square gives us

$$(y+3)^2=14$$
$$y=\pm\sqrt{14}-3$$
$$y^2=23\mp 6\sqrt{14}$$

Since $y^2>4$ only when $y=-\sqrt{14}-3$, the sum of the real solutions is $-3-\sqrt{14}$, so $|a|+|b|+|c|=|-3|+|-1|+|14|=\boxed{18}$.

14. (8 pts) How many distinct ways are there to distribute 15 identical candies to 4 distinct children—Alice, Bob, Charlie, and David—subject to the following conditions:

    i. Alice must receive at least 1 candy.
    ii. Bob must receive at most 3 candies.
    iii. Charlie must receive an even number of candies (zero is considered an even number).
    iv. David must receive an odd number of candies.

    Shubham Patel

    Solution Alice, Bob, Charlie, David receive A, B, C, D candies, respectively. By the conditions:

$$A \geq 1, \quad 0 \leq B \leq 3, \quad C = 2c,\ c \geq 0, \quad D = 2d + 1,\ d \geq 0,$$

and

$$A + B + 2c + (2d + 1) = 15 \implies A + B + 2(c + d) = 14.$$

Set $A' = A - 1 \geq 0$ and $t = c + d \geq 0$. Then

$$A' + B + 2t = 13,$$

with $B \in \{0, 1, 2, 3\}$. For each fixed t:

- $\#\{(c, d) : c + d = t\} = t + 1.$
- $\#\{(A', B) : A' + B = 13 - 2t,\ 0 \leq B \leq 3\} = \min(3,\ 13 - 2t) + 1.$

Hence the total number is

$$\sum_{t=0}^{6}(t+1)\big(\min(3, 13-2t)+1\big) = \sum_{t=0}^{5}(t+1)\cdot 4 \ +\ (6+1)\cdot 2 = 4\sum_{k=1}^{6} k + 14 = 4 \cdot 21 + 14 = \boxed{98}.$$

15. (8 pts) Points A, B, C, and D all lie on a circle. Chords AC and BD intersect at point E. Given that $BC = 6$, $AE = 4\sqrt{3}$, and $\angle BAC = \angle BCA = 30^\circ$, find the length of AD.

    Ryan Chan

    Solution Since $\angle BAC = \angle BCA$, $\triangle ABC$ is isosceles. Therefore, $AB = BC = 6$. We also know that $\angle ADB = \angle ACB = 30^\circ$ because they both subtend arc AB. Since

$$\frac{AE}{AB} = \frac{4\sqrt{3}}{6} = \frac{2}{\sqrt{3}},$$

$\triangle BAE$ must be a 30-60-90 triangle. Therefore, $\triangle ABD$ is also a 30-60-90 triangle, so

$$AD = 2AB = 2 \cdot 6 = \boxed{12}.$$

16. (8 pts) Let S be the set of all tuples of positive integers (a, b, c), such that $abc = 2^3 \cdot 3^3 \cdot 5^3$. Find the remainder of

$$\sum_{(a,b,c)\in S} (a+b+c)$$

when divided by 1000.

Sujith Miriyala

Solution By symmetry, we have that the given expression is equal to

$$3\sum_{a\in S} a$$

Let $a = 2^i 3^j 5^k$, where $0 \le i, j, k \le 3$. Notice that each $2^i 3^j 5^k$ is counted $(4-i)(4-j)(4-k)$ times.

We can now write the expression as

$$3\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{k=0}^{3} 2^i 3^j 5^k (4-i)(4-j)(4-k) \implies 3\sum_{i=0}^{2} 3^i(4-i)\sum_{j=0}^{3} 3^j(4-j)\sum_{k=0}^{3} 5^k(4-k)$$

This yields $3(4+6+8+8)(4+9+18+27)(4+15+50+125) = 3\cdot 26\cdot 58\cdot 194 = 3\cdot 1508\cdot 194$. Taking $\pmod{1000}$ gives $\boxed{656}$.

17. (9 pts) Real numbers a, b, c, where $a < b < c$, satisfy the following system

$$a^2bc + ab^2c + abc^2 = 90$$

$$3abc + a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 = \frac{255}{2}$$

$$a^2b^2c + ab^2c^2 + a^2bc^2 = 204$$

Find $2a + b + c$.

Sujith Miriyala

Solution The system factors into

$$abc(a + b + c) = 90$$

$$(ab + bc + ac)(a + b + c) = \frac{255}{2}$$

$$abc(ab + bc + ac) = 204$$

Let $u = a + b + c$, $v = ab + bc + ac$, and $w = abc$

We have that

$$uw = 90$$

$$uv = \frac{255}{2}$$

$$vw = 204$$

multiplying the equations together yields $u^2v^2w^2 = 15^2 \cdot 17^2 \cdot 6^2 \implies uvw = 15 \cdot 17 \cdot 6$.

From here, it is easy to see that $v = 17$, $w = 12$, and $u = \frac{15}{2}$, Or that

$$a + b + c = \frac{15}{2}$$

$$ab + bc + ac = 17$$

$$abc = 12$$

Suppose a, b, c are the roots of a cubic polynomial. Using Vieta's formulas, we find that this polynomial is $x^3 - \frac{15}{2}x^2 + 17x - 12$. By testing roots, and using polynomial division, we factor this cubic into $(x-2)(x-4)(x-\frac{3}{2}) \implies a = \frac{3}{2}, b = 2, c = 4$. Thus, $2a+b+c = 3+2+4 = \boxed{9}$.

18. (9 pts) A standard deck of 52 playing cards contains 4 Kings and 4 Queens. The expected number of cards between the topmost King and topmost Queen can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m+n$.

Aryan Raj

Solution Let's only worry about the 4 Kings and the 4 Queens.

Then, each of the 9 gaps have an EV of $\frac{52-8}{9} = \frac{44}{9}$ cards.

WLOG let the topmost King be above the topmost Queen.

KQXXXXXX gives $\frac{44}{9}$

KKQXXXXX gives $1 + 2 \cdot \frac{44}{9}$

KKKQXXXX gives $2 + 3 \cdot \frac{44}{9}$

KKKKQQQQ gives $3 + 4 \cdot \frac{44}{9}$

Therefore, the final EV is

$$\frac{20(\frac{44}{9}) + 10(1 + 2 \cdot \frac{44}{9}) + 4(2 + 3 \cdot \frac{44}{9}) + 1(3 + 4 \cdot \frac{44}{9})}{35} = \frac{379}{45},$$

so the answer is $379 + 45 = \boxed{424}$.

19. (9 pts) Let A, B, C, D, E, F, and O be points in the plane. Line AB is tangent to a circle with center O and B is on the circle. Line AD passes through point O and intersects the circle at points C and D. Line AF intersects the circle at points E and F. Given that $AE = 5$, $EF = 2.2$, $AD > AC$, $AF > AE$, and the radius of the circle is 2.5, find $AB + AC$.

Ryan Chan

Solution Call the circle ω. By Power of a Point Theorem,

$$\text{Pow}_\omega(A) = AB^2 = AC \cdot AD = AE \cdot AF = 5(5 + 2.2) = 36.$$

Therefore, $AB = 6$ and

$$AC \cdot AD = AC(AC + 2 \cdot 2.5) = AC(AC + 5) = 36,$$

so $AC = 4$. Therefore, the answer is $AB + AC = 6 + 4 = \boxed{10}$.

20. (9 pts) Let b be the smallest positive multiple of 2025 whose digits are all 0 or 1. Compute the decimal value of b interpreted as a binary number.

For example, if $b = 10110$, then your answer should be $2^4 + 2^2 + 2^1 = 22$.

Aryan Raj

Solution b can't end in 5, so it must be a multiple of 4050. b can't end in 50, so it must be a multiple of 8100.

Therefore, it suffices to compute the smallest positive multiple of 81 whose digits are all 0 or 1. Let us compute $10^x \mod 81$ for various x.

$$1, 10, 19, 28, 37, 46, 55, 64, 73, 1, 10, 19, \ldots$$

Note that this pattern holds because $10a \equiv a + 9 \pmod{81}$ for $a \equiv 1 \pmod 9$.

Also recall that any multiple of 9 must have sum of digits divisible by 9.

However, $111111111 \equiv 1 + 10 + 19 + 28 + 37 + 46 + 55 + 64 + 73 \equiv 9 \pmod{81}$.

Therefore, the smallest desired multiple of 81 must have at least 10 digits.

Adding $10^9 \equiv 1 \pmod{81}$ means we must subtract $10^1 \equiv 10 \pmod{81}$.

Therefore, the smallest desired multiple of 81 is 1111111101.

Our answer is 111111110100 converted from binary to decimal, which is

$$4(2^{10} - 1 - 2) = 4 \cdot 1021 = \boxed{4084}.$$

21. (12 pts) Let the roots of $x^2 + ax + b = 0$ be r and s where $r \neq s$. If $|r - s| = b$ and $a + b = r + s$, then $|ab|$ can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Aryan Raj

Solution $x^2 + ax + b = (x - r)(x - s)$ so $r + s = -a$ and $rs = b$. Since $a + b = r + s$, this means that $a + b = -a$ so $2a + b = 0$.

Therefore, r and s are now the roots of $x^2 + ax - 2a = 0$, or $(x + \frac{a}{2})^2 = \frac{a^2}{4} + 2a$.

Therefore, $b = |r - s| = |2\sqrt{\frac{a^2}{4} + 2a}| = |\sqrt{a^2 + 8a}|$, so $b^2 = |a^2 + 8a|$.

Since $b = -2a$, this means that $4a^2 = |a^2 + 8a|$, so $a = \frac{8}{3}$ or $a = -\frac{8}{5}$.

Since $b = |r - s| \geq 0$, we must have $a \leq 0$, so $a = -\frac{8}{5}$, and $b = \frac{16}{5}$.

Therefore, $|ab| = \frac{128}{25}$ and the answer is $128 + 25 = \boxed{153}$.

22. (12 pts) Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let (ABC) be the circumcircle of $\triangle ABC$. Let T be on BC such that $AT \perp BC$. Let I be the midpoint of BC. Let $Y \neq A$ be an intersection of AT and (ABC). Let $L \neq A$ be an intersection of AI and (ABC). If the area of quadrilateral $TILY$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$.

Aryan Raj

Solution Through either Pythagorean Triples or Heron's Formula, $BT = 5$, $TI = 2$, $IC = 7$, and $AT = 12$. By Parallelogram Law, $2AI^2 + 7 \cdot 14 = 13^2 + 15^2$ so $AI = 2\sqrt{37}$. By Power of a Point Theorem, $12TY = AT \cdot TY = BT \cdot TC = 45$ so $TY = \frac{15}{4}$ so $AY = 12 + \frac{15}{4} = \frac{63}{4}$ and $2IL\sqrt{37} = AI \cdot IL = BI \cdot IC = 49$ so $IL = \frac{49}{2\sqrt{37}}$ so $AL = 2\sqrt{37} + \frac{49}{2\sqrt{37}} = \frac{197}{2\sqrt{37}}$. Notice that

$$[TILY] = [ALY] - [ATI] = \frac{21}{16} \cdot \frac{197}{148} \cdot [ATI] - [ATI] = \frac{1769}{2368} \cdot [ATI] = \frac{1769}{2368} \cdot 12 = \frac{5307}{592}$$

so our answer is $5307 + 592 = \boxed{5899}$.

23. (12 pts) Let $r(n)$ be the number formed when the digits of n are reversed. Compute the sum of all prime numbers p such that $p < 500$ and $p + r(p)$ is a perfect square.

Shubham Patel

Solution When p has 1 digit, $p + r(p) = 2p$, which is only a perfect square for $p = 2$.

When p has 2 digits, let $p = 10a + b$ where $0 < a, b < 10$, so $p + r(p) = 11(a + b)$. This is only a perfect square for $a + b = 11$, so we get $p = 29, 47, 83$ as solutions.

When p has 3 digits, let $p = 100a + 10b + c$ where $0 < a, b, c < 10$, so $p + r(p) = 101(a + c) + 20b$. Note that all perfect squares are in $\{0, 1\} \pmod 4$ and $\{0, 1, 4\} \pmod 5$. By Chinese Remainder Theorem, this means that all perfect squares are in $\{0, 1, 4, 5, 9, 16\} \pmod{20}$. Since $1 + 1 \leq a + c \leq 4 + 9 = 13$, we get that $a + c \in \{4, 5, 9\}$. When $a + c = b$, we get $p = 100a + 10(a + c) + c = 110a + 11c$, which is divisible by 11, so $a + c \neq b$.

When $a + c = 4$, we only get $b = 4$, which is bad because then $a + c = b$.

When $a + c = 5$, we only get $b = 6$, giving solutions $p = 263, 461$.

When $a + c = 9$, we only get $b = 9$, which is bad because then $a + c = b$.

Therefore, the answer is $2 + 29 + 47 + 83 + 263 + 461 = \boxed{885}$.

24. (12 pts) Let $f(x)$ denote the number of permutations of the set $\{1, 2, \ldots, x\}$ such that for every position n where $1 \leq n \leq x$, the element at position n is either 1 or divisible by n. Compute $|f(20) - f(25)|$.

Aryan Raj

Solution Say that we put 1 in the a_0th term. Let a_k be in the a_{k+1}th term for all k. Then, we will have a be a periodic sequence with some period p, so $a_{p-1} = 1$.

Notice that $1|a_{p-2}| \ldots |a_1|a_0$ by definition. Call the tuple $(1, a_{p-2}, \ldots, a_1, a_0)$ a block.

$f(x)$ counts the number of blocks such that $a_0 \leq x$. Let $g(x)$ be the number of blocks such that $a_0 = x$. Then, $f(x) = g(1) + g(2) + \cdots + g(x)$, and $g(x) = \sum_{d|x, d<x} g(d)$.

Therefore, $f(25) - f(20) = g(21) + g(22) + g(23) + g(24) + g(25)$.

$g(1) = 1$

$g(q) = g(1) = 1$ for all primes q

$g(21) = g(1) + g(3) + g(7) = 1 + 1 + 1 = 3$

$g(22) = g(1) + g(2) + g(11) = 1 + 1 + 1 = 3$

$g(23) = 1$

$g(25) = g(1) + g(5) = 2$

Therefore, we wish to compute $9 + g(24)$.

$g(4) = g(1) + g(2) = 1 + 1 = 2$

$g(6) = g(1) + g(2) + g(3) = 1 + 1 + 1 = 3$

$g(8) = g(1) + g(2) + g(4) = 1 + 1 + 2 = 4$

$g(12) = g(1) + g(2) + g(3) + g(4) + g(6) = 1 + 1 + 1 + 2 + 3 = 8$

$g(24) = g(1) + g(2) + g(3) + g(4) + g(6) + g(8) + g(12) = 1 + 1 + 1 + 2 + 3 + 4 + 8 = 20$

Therefore, $9 + g(24) = \boxed{29}$.

25. (12 pts) Compute the number of functions $f : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, \ldots, 2024\}$ such that

- $f(i) \mid f(j)$ whenever $i \mid j$, and
- $f(x) \mid 2024$ for all $x \in \{1, 2, 3, 4, 5, 6\}$,

where $a \mid b$ if and only if a is a divisor of b.

Aryan Raj

Solution Consider some prime p and figuring out v_p of f. Note that the only (i, j) that matter are $(1, 2), (1, 3), (1, 5), (2, 4), (2, 6), (3, 6)$.

If $v_p(f(1)) = v_p(2024)$ then there is 1 way.

If $v_p(f(1)) = v_p(2024) - 1$ then there are $2(2 + 2 \cdot 3) = 16$ ways.

If $v_p(f(1)) = v_p(2024) - 2$ then there are $3(3 + 2 \cdot 5 + 3 \cdot 6) = 93$ ways.

If $v_p(f(1)) = v_p(2024) - 3$ then there are $4(4 + 2 \cdot 7 + 3 \cdot 9 + 4 \cdot 10) = 340$ ways.

Since $2024 = 2^3 \cdot 11 \cdot 23$, our answer is $(1 + 16 + 93 + 340)(1 + 16)(1 + 16) = \boxed{130050}$.

TB. (Tiebreak) Nischal has 2025 squares in a row. He wants to color each of them red, green, or blue. Estimate the sum of the digits of the number of colorings where no 3 consecutive squares are the same color.

Shubham Patel

Solution The answer is $\boxed{4083}$. Read below to see how to make an accurate estimate.

Let $a(n)$ be the answer when there are n squares instead of 2025 squares.

If the $n-1$th and nth squares are the same color, there are $a(n-2)$ ways to color the first $n-2$ squares, and 2 ways to color the $n-1$th and nth squares.

If the $n-1$th and nth squares are different colors, there are $a(n-1)$ ways to color the first $n-1$ squares, and 2 ways to color the nth square.

Therefore, $a(n) = 2a(n-1) + 2a(n-2)$.

If

$$A(x) = \sum_{n=0}^{\infty} a(n)x^n,$$

then this means that

$$A(x) = (2x + 2x^2)A(x) + \frac{3}{2},$$

so

$$A(x) = \frac{\frac{3}{2}}{1 - 2x - 2x^2} = \frac{\frac{3+\sqrt{3}}{4}}{1-(1+\sqrt{3})x} + \frac{\frac{3-\sqrt{3}}{4}}{1-(1-\sqrt{3})x}.$$

This means that,

$$a(n) = \frac{3+\sqrt{3}}{4}(1+\sqrt{3})^n + \frac{3-\sqrt{3}}{4}(1-\sqrt{3})^n \approx \frac{3+\sqrt{3}}{4}(1+\sqrt{3})^n.$$

Estimating logarithms can provide a good estimate for the number of digits of $a(2025)$, after which multiplying by $\frac{0+9}{2} = 4.5$ provides a good estimate for the sum of the digits of $a(2025)$.

For reference, $a(2025) =$ 917808118493726568895820944520127200653634968068853703533634576084510254734192634207409183715492603429742822695400665487852488949950795539267451724739405719111610226946564532925160904365615966315335896530702159276990086952374355258305480719038858991756109515182844006836691123902875906673270110393978867250869075769998500380838861275622062344372855877152595697088991956576303870902095518445225715751387535368837952916467923789417398931266174162733182715825906334885092217842425333839697036444651722263387145550767404066385487877516807289529093818006004548446181889763000699063254723864963609849494578169138600731311982237898393495546145182550284543444119791823485080216549965388471444141299403254076436332900146777140267723287800277143899730070264464329899049461142407402176623577817858842488638991117548061476729095104894281371058779927113631966245734587491823311787935809141022392\u200b32.